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NON-GAUSSIAN NON-HERMITEAN RANDOM MATRIX THEORY: phase transition and addition formalism

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Abstract

We apply the recently introduced method of hermitization to study in the large N limit non-hermitean random matrices that are drawn from a large class of circularly symmetric non-Gaussian probability distributions, thus extending the recent Gaussian non-hermitean literature. We develop the general formalism for calculating the Green's function and averaged density of eigenvalues, which may be thought of as the non-hermitean analog of the method due to Brèzin, Itzykson, Parisi and Zuber for analyzing hermitean non-Gaussian random matrices. We obtain an explicit algebraic equation for the integrated density of eigenvalues. A somewhat surprising result of that equation is that the shape of the eigenvalue distribution in the complex plane is either a disk or an annulus. As a concrete example, we analyze the quartic ensemble and study the phase transition from a disk shaped eigenvalue distribution to an annular distribution. Finally, we apply the method of hermitization to develop the addition formalism for free non-hermitean random variables. We use this formalism to state and prove a non-abelian non-hermitean version of the central limit theorem.

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1 Introduction

There has been considerable interest in random non-hermitean matrices recently. Possible applications range over several areas of physics[1, 2]. One difficulty is that the eigenvalues of non-hermitean matrices invade the complex plane, and consequently, various methods developed over the years to deal with random hermitean matrices are no longer applicable, as these methods typically all involve exploiting the powerful constraints of analytic function theory. (See in particular the paper by Brézin, Itzykson, Parisi, and Zuber[3].) In a recent paper[4], we proposed a “method of hermitization”, whereby a problem involving random non-hermitean matrices can be reduced to a problem involving random hermitean matrices, to which various standard methods (such as the diagrammatic method[5], or the “renormalization group” method[6, 7, 8, 9]) can be applied.

To our knowledge, the literature on random non-hermitean matrices[1, 2] has focussed exclusively on Gaussian randomness. For instance, it has been known for over thirty years, from the work of Ginibre[10], that for the Gaussian probability distribution $P(\phi) = (1/Z)\exp(-N\text{tr}\phi^\dagger\phi)$ (here, as in the rest of this paper, ϕ denotes an $N \times N$ random matrix with the limit $N \rightarrow \infty$ understood), the density of eigenvalues of ϕ is uniformly distributed over a disk of radius 1 in the complex plane. In this paper, we point out that using the method of hermitization we can determine the density of eigenvalues of probability distribution of the form

$$P(\phi) = \frac{1}{Z} e^{-N\text{tr}V(\phi^\dagger\phi)}, \quad (1.1)$$

where V is an arbitrary polynomial of its argument. Indeed, by a simple trick, we show that we can obtain the desired density of eigenvalues with a minimal amount of work, by judiciously exploiting the existing literature on random hermitean matrices.

Due to the symmetry of $P(\phi)$ under the transformation $\phi \rightarrow e^{i\alpha}\phi$, the density of eigenvalues is obviously rotational invariant. We find that the class of probability distributions of the form (1.1) exhibits a universal behavior in the sense that whatever the polynomial V is, the shape of the eigenvalue distribution in the complex plane is

always either a disk or an annulus.

In a certain sense, our work may be thought of as the analog of the work of Brézin et al. for random hermitean matrices [3]; they showed how the density of eigenvalues of hermitean matrices φ taken from the probability distribution $P(\varphi) = (1/Z)\exp[-N\text{tr}V(\varphi)]$ with V an arbitrary polynomial can be determined, and not just for the Gaussian case studied by Wigner and others[11], in which $V = (1/2)\text{tr}\varphi^2$. An important simplifying feature of the analysis in [3] is that $P(\varphi)$ depends only on the eigenvalues of φ , and not on the unitary matrix that diagonalizes it. In contrast, the probability distribution (1.1) for non-hermitean matrices depends explicitly on the $GL(N)$ matrix S used to diagonalize $\phi = S^{-1}\Lambda S$, and S does not decouple. Remarkably however, for the Gaussian $P(\phi)$, Ginibre [10] managed to integrate over S explicitly and derived an explicit expression for the probability distribution of the eigenvalues of ϕ . Unfortunately, it is not clear how to integrate over S and derive the expression for the eigenvalue probability distribution for non-Gaussian distributions of the form (1.1). In this paper we circumvent this difficulty by using the method of hermitization.

As an explicit application of our method, we study the case $V(\phi^\dagger\phi) = m^2\phi^\dagger\phi + \frac{g}{2}(\phi^\dagger\phi)^2$ in detail. For m^2 positive, we expect we would get a disk-like distribution generalizing Ginibre's work. As we make $m^2 \equiv -\mu^2$ more and more negative, we expect a phase transition at some critical value μ_c^2 , after which we might imagine the disk fragmenting into an annulus. Indeed, we find the critical value

$$\mu_c^2 = \sqrt{2g}.$$

We also calculate the density of eigenvalues inside the annulus in detail.

The problem of adding random hermitean matrices has been much discussed in the recent literature[12, 5, 13, 14, 15, 16]. In particular, a Feynman diagrammatic proof of the formalism, written in terms of the so-called Blue's function, was given in [14]. In a series of very interesting papers[2], Zahed and his collaborators have extended the addition formalism to random non-hermitean matrices. It turns out that the formalism we used above to determine the density of eigenvalues of random non-

hermitean matrices can be naturally used to study the addition formalism. Thus, we obtain, in a way which we naturally think is quite transparent, the addition formalism for random non-hermitean matrices, thus reproducing the result of Zahed et al. We use this addition formalism to state and prove a non-abelian non-hermitean version of the central limit theorem.

2 The Two Dimensional Gas of Eigenvalues

As is well known, the Dyson gas approach represents one of the most powerful methods in analyzing hermitean random matrices. Diagonalizing a hermitean matrix $\phi = S^\dagger \Lambda S$ one writes

$$\int d\phi f(\phi) = \int_{U(N)} d\mu(S) \int \prod_k d\lambda_k \left(\prod_{i < j} (\lambda_i - \lambda_j)^2 \right) f(\lambda_k, S). \quad (2.1)$$

For non-hermitean random matrices, one can diagonalize $\phi = S^{-1} \Lambda S$ and write

$$\int d\phi d\phi^\dagger f(\phi, \phi^\dagger) = \int_{GL(N)} d\mu(S) \prod_k d\lambda_k d\lambda_k^* \left(\prod_{i < j} |\lambda_i - \lambda_j|^4 \right) f(\lambda_k, \lambda_k^*, S). \quad (2.2)$$

The fourth power in $(\lambda_i - \lambda_j)$ here, in contrast to the second power in (2.1), is easily understood by dimensional analysis: a non-hermitean complex matrix has twice as many real variables as a hermitean matrix. The important difference between the hermitean and the non-hermitean cases is already apparent in the Gaussian case considered by Ginibre. The probability distribution

$$P(\phi^\dagger \phi) = \frac{1}{Z} e^{-N \text{tr } \phi^\dagger \phi} = \frac{1}{Z} e^{-N \text{tr } [SS^\dagger \Lambda^* (SS^\dagger)^{-1} \Lambda]} \quad (2.3)$$

depends on S . In the case of hermitean matrices, S would be unitary and hence disappear from (2.3) entirely. Here, however, one must integrate over S . After some work, Ginibre [10] showed that

$$\int dS e^{-N \text{tr} [SS^\dagger \Lambda^* (SS^\dagger)^{-1} \Lambda]} = \text{constant} \prod_{i < j} |\lambda_i - \lambda_j|^{-2} e^{-N \sum_k |\lambda_k|^2} \quad (2.4)$$

and was thus able to proceed with the gas approach. Once again, the problem reduces to the statistical mechanics of a gas, this time in two dimensions, with a logarithmic repulsion between the gas molecules and confined by a harmonic potential. Following Brézin et al. [3] in evaluating the integral by steepest descent, we obtain

$$\vec{x} = \int d^2 y \, \rho(\vec{y}) \frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|^2}.$$

Making an analogy with two dimensional electrostatics mentioned, the solution of this equation is immediate: ρ is equal to $1/\pi$ inside a disk of radius 1 centered at the origin, and vanishes outside the disk.

We see however, that had we been faced with a more complicated probability distribution, of the form $P(\phi) = \frac{1}{Z} e^{-N \text{tr } V(\phi^\dagger \phi)} = \frac{1}{Z} e^{-N \text{tr } V[SS^\dagger \Lambda^* (SS^\dagger)^{-1} \Lambda]}$, we do not know how to carry out the integration over S . In this paper we circumvent this difficulty and present an alternative way of calculating the density of eigenvalues of ϕ , based on the method of hermitization. It is an interesting challenge to obtain our results presented below using the gas method, and we conjecture to that end, that in the non-Gaussian case there should be a closed form expression for the integral over S that is analogous to (2.4).

3 The Method of Hermitization

Here we briefly review the discussion in our recent paper[4]. The averaged density of eigenvalues

$$\rho(x, y) = \langle \frac{1}{N} \sum_i \delta(x - \text{Re } \lambda_i) \delta(y - \text{Im } \lambda_i) \rangle \quad (3.1)$$

of the non-hermitean matrix ϕ , may be determined from the following two alternative quantities. The first quantity is

$$F(z, z^*) = \langle \frac{1}{N} \text{tr} \log (z - \phi)(z^* - \phi^\dagger) \rangle, \quad (3.2)$$

in terms of which¹

$$\rho(x, y) = \frac{1}{\pi} \partial \partial^* F(z, z^*). \quad (3.3)$$

The other quantity is the Green's function associated with ϕ , namely

$$G(z, z^*) = \langle \frac{1}{N} \text{tr} \frac{1}{z - \phi} \rangle = \int d^2 x' \frac{\rho(x', y')}{z - z'}, \quad (3.4)$$

in terms of which

$$\rho(x, y) = \frac{1}{\pi} \partial^* G(z, z^*). \quad (3.5)$$

The probability distributions (1.1) studied in this paper are invariant under $\phi \rightarrow e^{i\alpha} \phi$, rendering

$$\rho(x, y) \equiv \rho(r)/2\pi \quad (3.6)$$

circularly invariant. Rotational invariance thus leads to a simpler form of the defining formula (3.4) for $G(z, z^*)$ which reads

$$\gamma(r) \equiv z G(z, z^*) = \int_0^r r' dr' \rho(r'), \quad (3.7)$$

whence

$$\rho(r) = \frac{1}{r} \frac{d\gamma}{dr}. \quad (3.8)$$

¹We use the following notational conventions: for $z = x + iy$ we define $\partial \equiv \frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$ so that $\partial z = 1$. Similarly, we define $\partial^* \equiv \frac{\partial}{\partial z^*} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$, so that $\partial^* z^* = 1$ and also $\partial^*(1/z) = \pi \delta^{(2)}(z)$. Finally, we denote $|z| = r$.

Clearly, the quantity $\gamma(r)$ is a positive monotonically increasing function, which satisfies the obvious “sum-rules”

$$\gamma(0) = 0 \quad \text{and} \quad \gamma(\infty) = 1. \quad (3.9)$$

In particular, observe that the first condition in (3.9) insures that no $\delta(x)\delta(y)$ spike arises in $\rho(x, y)$ when calculating it from (3.5) with $G(z, z^*)$ given by (3.7), as it should be.

We have mentioned in the introduction that a straightforward diagrammatic method is not allowed for non-hermitean matrices. The method of hermitization[4] enables us to arrive at a diagrammatic method indirectly by reducing the problem of determining the eigenvalue density of random non-hermitean matrices to the more elementary problem of determining the eigenvalue density of random hermitean matrices (for which the diagrammatic method may be applied.) To make the paper self-contained, we summarize the method of hermitization here in the form of an algorithm.

Given a set of random non-hermitean $N \times N$ matrices ϕ , construct the auxiliary set of random hermitean $2N \times 2N$ matrices

$$H = \begin{pmatrix} 0 & \phi - z \\ \phi^\dagger - z^* & 0 \end{pmatrix}. \quad (3.10)$$

Then one applies one’s favorite method of hermitean random matrix theory to calculate the propagator associated with H , namely,

$$\mathcal{G}_\nu^\mu(\eta; z, z^*) = \left\langle \left(\frac{1}{\eta - H} \right)_\nu^\mu \right\rangle, \quad (3.11)$$

where η is a complex variable and the indices μ and ν run over all possible $2N$ values. In particular, in the diagrammatic method, one simply expands \mathcal{G}_ν^μ in powers of $1/\eta$, with interaction vertices H . This is a well defined procedure for large η , and it converges to a function which is analytic in the complex η plane, except for the cut (or cuts) along the real axis which contain the eigenvalues of H . After summing this

series (and thus determining $\mathcal{G}_\nu^\mu(\eta; z, z^*)$ in closed form), we are allowed to set $\eta \rightarrow 0$ in (3.11).²

As mentioned, we can now calculate $\rho(x, y)$ by two different methods, each with its advantages. The first is to simply observe that

$$-\frac{1}{H} = \begin{pmatrix} 0 & \frac{1}{z^* - \phi^\dagger} \\ \frac{1}{z - \phi} & 0 \end{pmatrix},$$

from which it follows that

$$G(z, z^*) = \frac{1}{N} \text{tr}_{(2N)} \left[\begin{pmatrix} 0 & \mathbf{1}_N \\ 0 & 0 \end{pmatrix} \mathcal{G}(0; z, z^*) \right], \quad (3.12)$$

enabling us to use (3.5) to express $\rho(x, y)$. This observation is the basis for many of the calculations in [2].

An alternative is to take the trace of \mathcal{G}_ν^μ :

$$\mathcal{G}(\eta; z, z^*) = \frac{1}{2N} \langle \text{tr}_{(2N)} \frac{1}{\eta - H} \rangle = \frac{\eta}{N} \langle \text{tr}_{(N)} \frac{1}{\eta^2 - (z^* - \phi^\dagger)(z - \phi)} \rangle, \quad (3.13)$$

from which one can construct an integral representation[4] of (3.2), thus enabling us to use (3.3) to determine $\rho(x, y)$. The determination of $\mathcal{G}(\eta; z, z^*)$ involves only the well-known and more elementary problem of determining the density of eigenvalues of the hermitean random matrices H , which is the essence of our method of hermitization.

Observe from (3.13) that

$$\mathcal{G}(0 - i0; z, z^*) = \frac{i\pi}{N} \langle \text{tr}_{(N)} \delta \left(\sqrt{(z^* - \phi^\dagger)(z - \phi)} \right) \rangle \quad (3.14)$$

and thus counts the (average) number of zero-eigenvalues of the positive semi-definite hermitean matrix $\sqrt{(z^* - \phi^\dagger)(z - \phi)}$ [4]. Thus, if (on the average) ϕ has no eigenvalues equal to z , or in other words, if the density of eigenvalues of ϕ vanishes at z , then

$$\mathcal{G}(0 - i0; z, z^*) = 0, \quad (3.15)$$

²Speaking colloquially, we may say that the crucial maneuver here is that while we cannot expand in powers of z , we can arrange for z to “hitch a ride” with η and at the end of the ride, throw η away.

independently of the large N limit. In particular, the boundaries of the eigenvalue distribution of ϕ in the large N limit, are the curve (or curves) in the complex plane which separate regions where $\mathcal{G}(0; z, z^*) = 0$ from regions where $\mathcal{G}(0; z, z^*) \neq 0$. As a matter of fact, one can infer the location of these boundaries even without an explicit knowledge of $\mathcal{G}(0; z, z^*)$ and $G(z, z^*)$, by investigating the “gap equation” for $\mathcal{G}(0; z, z^*)$, namely, the trace of both sides of (3.11), and then setting $\eta = 0$. The “gap equation” is a polynomial in \mathcal{G} with real coefficients (which depend on z explicitly as well as through $G(z, z^*)$, and the parameters of the potential V .) Due to the chiral nature of H , this polynomial contains a trivial factor of $\mathcal{G}(0; z, z^*)$ which we immediately factor out. Setting $\mathcal{G} = 0$ in the remaining factor we obtain an equation for the boundary. This is discussed in detail in [4].

To summarize, the method of hermitization allows us to reduce the problem of dealing with random non-hermitean matrices to the well-studied problem of dealing with random hermitean matrices. Given the non-hermitean matrix ϕ , we study the hermitean matrix H instead. By whatever method one prefers, once one has determined the quark propagator \mathcal{G}_ν^μ (or its trace, the Green’s function $\mathcal{G}(\eta; z, z^*)$), one can in principle obtain $\rho(x, y)$. Whether that can be done in practice is of course another story.

It is worth emphasizing that the formalism described here has nothing to do with large N as such. The formalism is also totally independent of the form of the probability distribution $P(\phi, \phi^\dagger)$.

4 Non-Gaussian Ensembles

Our task in this paper is to solve the more problem of determining the Green's function and the averaged eigenvalue density of a non-hermitean random matrix ϕ . Here we show that by applying a simple trick, we can obtain the desired density of eigenvalues with a minimal amount of work, by judiciously exploiting the existing literature on random hermitean matrices.

In this section we will follow a common practice and borrow some terminology from quantum chromodynamics: we may consider ϕ, ϕ^\dagger as “gluons” (in zero space-time dimensions), which interact with a $2N$ dimensional multiplet of “quarks” ψ^μ , with a complex mass matrix (the “inverse propagator”)

$$\mathcal{G}_0^{-1} = \begin{pmatrix} \eta & z \\ z^* & \eta \end{pmatrix} \quad (4.1)$$

(expressed in terms of its $N \times N$ blocks.) The matrix Green's function $\mathcal{G}_\nu^\mu = \langle (1/(\eta - H))_\nu^\mu \rangle$, in (3.11), which represents the quark propagator, is given by the Dyson-Schwinger equation in terms of the one-quark irreducible self energy Σ_ν^μ :

$$\mathcal{G}_\nu^\mu = \left(\frac{1}{\mathcal{G}_0^{-1} - \Sigma} \right)_\nu^\mu, \quad (4.2)$$

We emphasize that this equation always holds, and practically amounts to the definition of the one-particle irreducible self energy. For probability distributions of the form $P(\phi) = (1/Z)e^{-N\text{tr}V(\phi^\dagger\phi)}$ in (1.1) it is easy to see that³

$$\Sigma_\nu^\mu = \Sigma(\eta; r)\delta_\nu^\mu \quad (4.3)$$

(where we recall that $r = |z|$.) Note that Σ depends on r and not separately on z and z^* because it may be expanded as an infinite sum of traces of \mathcal{G}_0 , each term of

³Split the $2N$ dimensional quark multiplet into two N dimensional “flavors”, $\psi = (u, d)$. Then note that the “quark-quark-gluon” interaction is $\psi^\dagger \begin{pmatrix} 0 & \phi \\ \phi^\dagger & 0 \end{pmatrix} \psi = u^\dagger \phi d + d^\dagger \phi^\dagger u$, and use the fact that the fermion lines are directed, *e.g.* a ϕ vertex always absorbs a d and emits a u . Then invoke “index democracy”.

which is a function only of η and r . Thus,

$$\mathcal{G}_\nu^\mu = \frac{1}{r^2 - (\eta - \Sigma)^2} \begin{pmatrix} \Sigma - \eta & z \\ z^* & \Sigma - \eta \end{pmatrix}, \quad (4.4)$$

leading to

$$\mathcal{G} \equiv \frac{1}{2N} \sum_\mu \mathcal{G}_\mu^\mu = \frac{\Sigma - \eta}{r^2 - (\eta - \Sigma)^2}. \quad (4.5)$$

The important point is that in the large N limit, the one-quark irreducible self energy Σ can be written in terms of the cumulants Γ_{2k} of $P(\phi) = (1/Z)e^{-N\text{tr}V(\phi^\dagger\phi)}$ (Eq. (1.1)), namely, the connected correlators involving k ϕ 's and ϕ^\dagger 's. Diagrammatically speaking, the Γ_{2k} may be thought of as a “blob” out of which emanate k ϕ 's and ϕ^\dagger 's, which cannot be separated into two smaller blobs, with k_1 ϕ 's and ϕ^\dagger 's in one blob and k_2 ϕ 's and ϕ^\dagger 's in the other (with $k_1 + k_2 = k$ of course.) We then obtain

$$\Sigma(\eta; r) = \sum_{k=1}^{\infty} \Gamma_{2k} [\mathcal{G}(\eta; r)]^{2k-1}, \quad (4.6)$$

in a fashion similar to the calculation in [14]. Recall also that for the Gaussian ensemble only the variance $\Gamma_2 \neq 0$, and thus simply $\Sigma = \Gamma_2 \mathcal{G}$.

If we knew Σ in terms of \mathcal{G} , then we can solve for \mathcal{G} and hence the density of eigenvalues. We are thus faced with the problem of determining the Γ_{2k} 's which appears to be a rather difficult task.

It is important to note that the Γ_{2k} in (4.6) depend only on the probability distribution $P(\phi)$ shown above and not on the particular quantity we average over. Consider the problem of determining the eigenvalue density of the hermitean matrix $\phi^\dagger\phi$. We would study the Green's function

$$F(w) = \left\langle \frac{1}{N} \text{tr}_{(N)} \frac{1}{w - \phi^\dagger\phi} \right\rangle \equiv \int_0^\infty \frac{\tilde{\rho}(\sigma) d\sigma}{w - \sigma}, \quad (4.7)$$

where $\tilde{\rho}(\mu)$ is the averaged eigenvalue density of $\phi^\dagger\phi$. We see from (3.13) that this Green's function is related to $\mathcal{G}(\eta; r=0)$ simply through

$$\mathcal{G}(\eta; r=0) = \eta F(\eta^2). \quad (4.8)$$

Also, for $z = 0$ we have from (4.5)

$$\Sigma(\eta; 0) = \eta - \mathcal{G}^{-1}(\eta; 0). \quad (4.9)$$

The crucial observation is that $F(w)$ is already known in the literature on chiral and rectangular block random hermitean matrices⁴ for the Gaussian distribution[15, 9, 17, 18], as well as for non-Gaussian probability distributions of the form (1.1) with an arbitrary polynomial potential $V(\phi^\dagger \phi)$ [19, 20]. In fact, the authors of [19, 20] simply calculated the diagonal elements of the propagator $\mathcal{G}_\nu^\mu(\eta; 0)$ using Dyson gas techniques (so the coefficients Γ_{2k} are only implicit in these papers.)

The contents of (4.6), (4.8) and (4.9) may thus be summarized as follows: there is a unique function $a(\xi)$ which behaves like $1/\xi$ as ξ tends to infinity, is regular at $\xi = 0$, and satisfies the equation $\xi - a^{-1} = \sum_{k=1}^{\infty} \Gamma_{2k} a^{2k-1}$. This function is

$$a(\xi) = \xi F(\xi^2) \equiv \mathcal{G}(\xi; 0). \quad (4.10)$$

We, on the other hand, are interested in the opposite case, where $\eta = 0, r = |z| \neq 0$. Consider the matrix $\mathcal{G}_\nu^\mu(0 - i0; z, z^*)$. Due to “color index democracy”, its $N \times N$ blocks are all proportional to the unit matrix $\mathbf{1}_N$, and we see that

$$\mathcal{G}_\nu^\mu(0 - i0; z, z^*) = \begin{pmatrix} \mathcal{G} & G^* \\ G & \mathcal{G} \end{pmatrix}, \quad (4.11)$$

where $\mathcal{G}(0 - i0; r)$ is given in (3.14) and the complex Green’s function $G(z, z^*)$ was defined⁵ in (3.12). Inverting the propagator in (4.11) and using (4.1) (with $\eta = 0$) and (4.2), we obtain the two equations

$$z = \frac{G^*}{|G|^2 - \mathcal{G}^2} \quad (4.12)$$

and

$$\Sigma = \frac{\mathcal{G}}{|G|^2 - \mathcal{G}^2}. \quad (4.13)$$

⁴Namely, matrices H with the block structure $H = \begin{pmatrix} 0 & \phi \\ \phi^\dagger & 0 \end{pmatrix}$.

⁵From this point on, unless otherwise stated, our notation will be such that $\mathcal{G} \equiv \mathcal{G}(0 - i0; r)$, $\Sigma \equiv \Sigma(0; r)$ and $G \equiv G(z, z^*)$. Note in particular from (3.14) that \mathcal{G} is then pure imaginary, with $\text{Im } \mathcal{G} \geq 0$.

Note from (4.12) that if $\mathcal{G} = 0$, then $G = 1/z$, which corresponds to the region outside the domain of the eigenvalue distribution[2, 4]. In addition, using (4.12) twice, we obtain

$$zG = \frac{|G|^2}{|G|^2 - \mathcal{G}^2} = (|G|^2 - \mathcal{G}^2) r^2. \quad (4.14)$$

Thus, due to the fact that \mathcal{G} is pure imaginary (see (3.14)) we conclude that $\gamma \equiv zG$ is always real and non-negative. Alternatively, we can write

$$\mathcal{G}^2 = \frac{zG(zG - 1)}{r^2}, \quad (4.15)$$

and therefore from the fact that $\mathcal{G}^2 \leq 0$ we conclude that $0 \leq \gamma \leq 1$, in accordance with the monotonicity of $\gamma(r)$ and the sum-rules $\gamma(0) = 0$ and $\gamma(\infty) = 1$ in (3.9).

Let us now define the pure imaginary quantity

$$\xi \equiv \frac{1}{\mathcal{G}} + \Sigma = \frac{1}{\mathcal{G}} + \sum_{k=1}^{\infty} \Gamma_{2k} \mathcal{G}^{2k-1}, \quad (4.16)$$

where the last equality follows from (4.6) (with $\eta = 0$). Using (4.12) and (4.13) we thus have simply

$$\xi = \frac{zG}{\mathcal{G}}. \quad (4.17)$$

It then follows from (4.10) and the statement preceding that equation, and from (4.17) that

$$\xi F(\xi^2) = \mathcal{G} = \frac{zG}{\xi}. \quad (4.18)$$

We have thus eliminated the Γ_{2k} !

From (4.15) and from (4.17) we determine $\xi = \gamma/\mathcal{G}$ to be

$$\xi^2 = \frac{\gamma}{\gamma - 1} r^2. \quad (4.19)$$

Note that $\xi^2 \leq 0$, consistent with the bounds $0 \leq \gamma \leq 1$. Comparing (4.18) and (4.8) we obtain the remarkable relation

$$\mathcal{G}(\xi; 0) = \mathcal{G}(0; r), \quad (4.20)$$

for ξ and r that are related by (4.19) (with ξ imaginary.)

We now have the desired equation for $\gamma \equiv zG$: substituting (4.19) into (4.18) we obtain

$$\gamma \left[r^2 F \left(\frac{\gamma r^2}{\gamma - 1} \right) - \gamma + 1 \right] = 0. \quad (4.21)$$

Thus, given F we can solve for $\gamma(r)$ using this equation.

From (4.7) we observe that for w large we may expand

$$F(w) = \frac{1}{w} + \frac{1}{w^2} f(w), \quad (4.22)$$

where $f(w) = f_0 + (f_1/w) + \dots$. Substituting $w = \xi^2 = \gamma r^2/(\gamma - 1)$ into (4.22) and letting γ tend to 1, (4.21) becomes

$$(\gamma - 1)^2 \left[\frac{f(w)}{\gamma r^2} - 1 \right], \quad (4.23)$$

and thus $\gamma = 1$, which corresponds to the region outside the eigenvalue distribution, is always a double root of (4.21).

We will now explain how the Green's function $F(w)$ is obtained in the literature[9, 19, 20]. Let V be a polynomial of degree p . From the definition of $F(w)$ in (4.7) and its analyticity property, we expect F , following the arguments of Brézin et al, to have the form⁶

$$F(w) = \frac{1}{2} V'(w) - P(w) \sqrt{(w - a)(w - b)}, \quad (4.24)$$

where

$$P(w) = \sum_{k=-1}^{p-2} a_k w^k. \quad (4.25)$$

The constants $0 \leq a < b$ and a_k are then determined completely by the requirement that $F(w) \rightarrow \frac{1}{w}$ as w tends to infinity, and by the condition that $F(w)$ has at most an integrable singularity as $w \rightarrow 0$.⁷

⁶Here we assume for simplicity that the eigenvalues of $\phi^\dagger \phi$ condense into a single segment $[a, b]$.

⁷Thus, if $a > 0$, inevitably $a_{-1} = 0$. However, if $a = 0$, then a_{-1} will be determined by the first condition.

5 Phase Transitions in the Quartic Ensemble

Having determined $F(w)$ in this way, we substitute it into (4.21) and find $G(z, z^*)$. We can thus calculate the density of eigenvalues explicitly for an arbitrary V . For the sake of simplicity we now focus on the quartic potential⁸

$$V(\phi^\dagger \phi) = 2m^2 \phi^\dagger \phi + g(\phi^\dagger \phi)^2. \quad (5.1)$$

5.1 The Disk Phase

For m^2 not too negative, which as we shall see translates here into $m^2 > -\sqrt{2g}$, we expect the density of eigenvalues to be a disk. In that case we expect that $\rho(0) > 0$, as in Ginibre's case. Thus, from (3.8) $\rho(r) = (1/r)(d\gamma/dr)$ and from the first sum rule $\gamma(0) = 0$ in (3.9) we conclude that⁹ $\gamma(r) \sim r^2$ near $r = 0$. Therefore, for r small, (4.21) yields $F(-r^4) \sim -1/r^2$, namely, $F(w) \sim 1/\sqrt{w}$ for $w \sim 0$, as we could have anticipated from Ginibre's case. This means that $a = 0$ in (4.24), and therefore,

$$F(w) = m^2 + gw - \left(\frac{a_{-1}}{w} + a_0 \right) \sqrt{w(w-b)}. \quad (5.2)$$

From the asymptotic behavior of $F(w)$ we then obtain¹⁰

$$a_0 = g, \quad c \equiv a_{-1} = \frac{2m^2 + \sqrt{m^4 + 6g}}{3}, \quad \text{and} \quad b = \frac{-2m^2 + 2\sqrt{m^4 + 6g}}{3g}. \quad (5.3)$$

For later convenience we record here that

$$bc^2 = 2 \frac{(m^4 + 6g)^{3/2} - m^2(m^4 - 18g)}{27g}. \quad (5.4)$$

Substituting (5.2) and (5.3) into (4.21) we arrive¹¹ at

$$\gamma^2 - \gamma [1 + 2(m^2 r^2 + g r^4)] + bc^2 r^2 = 0 \quad (5.5)$$

⁸In the Gaussian case, $V = \phi^\dagger \phi$, we have $2\xi F(\xi^2) = \xi - \sqrt{\xi^2 - 4}$, whence the roots of (4.21) are $\gamma = 0, 1$ and r^2 . We note that $\gamma = 0$ is unphysical, $\gamma = r^2$ (*i.e.*, $G = z^*$) corresponds to Ginibre's disk[10], and $\gamma = 1$ is the solution outside the disk.

⁹Alternatively, we can reach the same conclusion about $\gamma(r)$ for r small by looking at \mathcal{G}^2 . According to (3.15), \mathcal{G}^2 must not vanish in some neighborhood of the origin. Neither can it diverge there. It thus follows from (4.15) that $\gamma(r) \sim r^2$ near $r = 0$.

¹⁰ b is positive by definition. The expression for b in (5.3) is the positive root (also for $m^2 < 0$) of the quadratic equation which determines b . b then determines c via $c = m^2 + (gb/2)$.

¹¹In deriving (5.5) we divided through by the trivial factor $(\gamma - 1)^2$ alluded to in (4.23).

with roots

$$\gamma_{\pm}(r) = \frac{1}{2} \left[1 + 2(m^2 r^2 + gr^4) \pm \sqrt{[1 + 2(m^2 r^2 + gr^4)]^2 - 4bc^2 r^2} \right]. \quad (5.6)$$

Let us assume that $m^2 > -\sqrt{2g}$. Under this condition the the discriminant of (5.5) is non-negative, and contains a factor $(r^2 - b/4)^2$. The root of (5.5) which satisfies the condition $\gamma(0) = 0$ (see (3.9)) is $\gamma_-(r)$ (whereas $\gamma_+(0) = 1$.) However, a subtlety arises, since $\gamma_-(r)$ is monotonically increasing (as it should be, by definition) only for $r^2 < b/4$. The root $\gamma_+(r)$ increases monotonically only for $r^2 > b/4$. Due to the factor $(r^2 - b/4)^2$ in the discriminant of (5.5), we observe that $\gamma'_-(\sqrt{b}/2-) = \gamma'_+(\sqrt{b}/2+)$, and thus γ_+ and γ_- may be glued smoothly into a single monotonically increasing function $\gamma(r)_{disk}$ as

$$\gamma(r)_{disk} = \begin{cases} \gamma_-(r) & , \quad r^2 < b/4 \\ \gamma_+(r) & , \quad r^2 > b/4. \end{cases} \quad (5.7)$$

The eigenvalue density is then given by (3.8) as

$$\pi\rho(x, y) = m^2 + 2gr^2 + [\text{sgn}(\frac{b}{4} - r^2)] \frac{bc^2 - (m^2 + 2gr^2)[1 + 2(m^2 r^2 + gr^4)]}{\sqrt{[1 + 2(m^2 r^2 + gr^4)]^2 - 4bc^2 r^2}}. \quad (5.8)$$

In particular, note that $\rho(0, 0) = bc^2/\pi \geq 0$.

We now determine the radius of the disk r_b , namely, the boundary of the eigenvalue distribution. In this case there is only one boundary, and thus the r dependent solution of (4.21) increases monotonically from $\gamma(0) = 0$ to $\gamma(r_b) = 1$, since all the eigenvalues are included within $r \leq r_b$. Thus, setting $\gamma(r) = 1$ in (5.5) and using (5.4) we obtain

$$r_b^2 = (bc^2 - 2m^2)/2g = 2 \frac{(m^4 + 6g)^{3/2} - m^2(m^4 + 9g)}{27g}. \quad (5.9)$$

It is possible to show that $r_b^2 > b/4$. We thus find that the density at the boundary is

$$\pi\rho_b = \left[2r_b^2 + \frac{2m^2}{g} - \frac{1}{2gr_b^2} \right]^{-1} \quad (5.10)$$

which can be shown to be always positive.

In the limit $g \rightarrow 0$ (with $m^2 > 0$, of course) c tends to m^2 and both r_b^2 and $b/4$ tend to $1/(2m^2)$. Therefore, (5.7) simplifies into

$$\gamma(r)_{disk} = \begin{cases} 2m^2 r^2 & , \quad r^2 < 1/(2m^2) \\ 1 & , \quad r^2 > 1/(2m^2) . \end{cases}$$

Thus, from (3.8) (or by taking $g \rightarrow 0$ in (5.8)) we recover Ginibre's uniform density $\rho(x, y) = 2m^2/\pi$ inside a disk of radius $1/(2m^2)$, consistent with the $g \rightarrow 0$ limit of (5.1).

In the strong coupling limit $g/m^4 \gg 1$ we observe that $c \sim \sqrt{2g/3}$ and $b \sim \sqrt{8/3g}$. Therefore, in this limit the disk shrinks like $r_b^2 \sim \sqrt{8/27g}$, which means that the density should blow up like $1/r_b^2$. We find indeed that in this limit $\rho_b \sim (12/5\pi)\sqrt{6g}$. We also find that in this limit $\rho(0, 0) \sim (2/\pi)(2/3)^{3/2}\sqrt{g} \sim (5/27)\rho_b < \rho_b$, and the density grows as r increases.

To summarize, for $m^2 > -\sqrt{2g}$, the $\gamma(r)$ corresponding to the ensemble (5.1) in the disk phase is thus given by (5.7) for $r \leq r_b$ and is equal to one for $r > r_b$.

When m^2 becomes too negative, that is, for $m^2 < -\sqrt{2g}$, the discriminant of (5.5) becomes negative for some range of values of r , rendering $\gamma_{\pm}(r)$ complex, which is unacceptable. This signals that the disk phase becomes unstable, and the systems undergoes a phase transition, as we discuss in the next section. The boundary of the disk phase in the $m^2 - g$ plane is thus given by $m^2 = -\sqrt{2g}$. Consider approaching this boundary from within the disk phase by setting $m^2 = -\sqrt{2g} + \delta$, with δ positive and small. Then, using (5.3), we find to first order in δ that $c = \delta/2$ and $b = 2\sqrt{(2/g)} - \delta/g$. In particular, at the phase boundary itself $c = 0$. Consequently, at the phase boundary (5.8) simplifies into

$$\rho(x, y) = \begin{cases} 0 & , \quad r^2 < 1/\sqrt{2g} \\ 2\sqrt{2g}(\sqrt{2g}r^2 - 1) & , \quad 1/\sqrt{2g} < r^2 < r_b^2 . \end{cases} \quad (5.11)$$

where $r_b^2 = \sqrt{2/g}$ from (5.9). Thus, as we decrease δ to zero, $\rho(x, y)$ becomes increasingly depleted inside the disk $r^2 < b(\delta)/4$, reaching complete depletion at $\delta = 0$, at which point the disk breaks into an annulus. Finally, for later reference, let us note

that at the phase boundary (5.2) reads

$$F(w) = -\sqrt{2g} + gw - g \sqrt{w \left(w - 2\sqrt{\frac{2}{g}} \right)}. \quad (5.12)$$

5.2 The Annular Phase

We saw that at the phase boundary $m^2 = -\sqrt{2g}$ the disk configuration of the previous sub-section broke into an annulus. We thus expect that for $m^2 < -\sqrt{2g}$ the stable eigenvalue distribution would be annular.

For convenience, let us switch notations according to $m^2 = -\mu^2$, and also write $\mu_c^2 = \sqrt{2g}$.

We denote the radii of the annulus boundaries by $r_1 < r_2$. In the inner void, $r < r_1$, we clearly have $\gamma(r) \equiv 0$. Thus, (4.21) and (4.24) imply $a_{-1}\sqrt{ab} = 0$. The annulus must therefore arise for $a_{-1} = 0$ (the other possible solution $a = 0, a_{-1} \neq 0$ leads to a disk configuration with $\gamma = 0$ only at $r = 0$, as we discussed in the previous sub-section.) Therefore,

$$F(w) = m^2 + gw - a_0 \sqrt{(w-a)(w-b)}. \quad (5.13)$$

From the asymptotic behavior of $F(w)$ we then obtain

$$a_0 = g, \quad a = \frac{\mu^2}{g} - \sqrt{\frac{2}{g}}, \quad \text{and} \quad b = \frac{\mu^2}{g} + \sqrt{\frac{2}{g}}. \quad (5.14)$$

We see that $a = (2/\mu_c^4)(\mu^2 - \mu_c^2)$ which is positive for $\mu^2 > \mu_c^2$, as it should be, by definition.

We now determine $\gamma(r)$ inside the annulus. Substituting (5.13) and (5.14) into (4.21) and factoring out all the factors ¹² of $(\gamma - 1)$, we obtain a linear equation for γ and find

$$\gamma(r)_{\text{annulus}} = 1 - 2\mu^2 r^2 + 2gr^4, \quad (5.15)$$

which must correspond to the region inside the annulus.

¹²There is one extra such factor as compared to the disk phase.

The $\gamma(r)$ configuration (5.15) has to match continuously to $\gamma = 0$ at $r = r_1$ and to $\gamma = 1$ at $r = r_2$. From the matching at $r = r_2$ we find that

$$r_2^2 = \frac{\mu^2}{g} = \frac{2\mu^2}{\mu_c^4}. \quad (5.16)$$

The matching at $r = r_1$ yields a quadratic equation for r_1^2 with roots

$$\frac{\mu^2 \pm \sqrt{\mu^4 - 2g}}{2g} = \frac{\mu^2 \pm \sqrt{\mu^4 - \mu_c^4}}{\mu_c^4}, \quad (5.17)$$

both of which are smaller than the expression in (5.16). In order to determine which of these roots corresponds to r_1^2 we recall that $\gamma(r)$ must interpolate monotonically between its two boundary values, namely, that $\gamma'(r) > 0$ for $r_1 < r < r_2$, which is nothing but the statement that $\rho(r) > 0$. Substituting (5.15) in (3.8) we find

$$\rho(x, y) = \frac{4g}{\pi} \left(r^2 - \frac{\mu^2}{\mu_c^4} \right) = \frac{2\mu_c^2}{\pi} \left(\mu_c^2 r^2 - \frac{\mu^2}{\mu_c^2} \right), \quad r_1 < r < r_2. \quad (5.18)$$

Note that $\rho(r)$ increases monotonically. The correct normalization of (5.18) is guaranteed by $\gamma(r_2) = 1$, as can be checked explicitly. Finally, positivity of (5.18) determines r_1^2 in (5.17) as

$$r_1^2 = \frac{\mu^2 + \sqrt{\mu^4 - \mu_c^4}}{\mu_c^4}. \quad (5.19)$$

Note that $r_1^2 = 1/\mu_c^2 = 1/\sqrt{2g}$ at $\mu^2 = \mu_c^2$, which coincides with the radius (squared) of the depleted region in the disk configuration (5.11). Also, at $\mu^2 = \mu_c^2$ we have $r_2^2 = 2/\mu_c^2 = \sqrt{2/g}$, which coincides with the disk radius squared r_b^2 in (5.9) at the phase boundary. Thus, at the phase boundary $\mu^2 = \mu_c^2$ (5.18) coincides with (5.11), namely, $\rho(x, y)$ is continuous at the transition from the disk phase to the annular phase.

Consider approaching the phase boundary $m^2 = -\sqrt{2g}$ from within the annular phase by setting $\mu^2 = \sqrt{2g} + \delta$, with δ positive and small. Then, since all the expressions in (5.14) are linear in μ^2 , we find that $a = \delta/g$ and $b = 2\sqrt{(2/g)} + \delta/g$. Thus, at the phase boundary itself $a = 0$, and $b = 1/\sqrt{2g}$. Therefore, at the phase boundary (5.13) reads

$$F(w) = -\sqrt{2g} + gw - g \sqrt{w \left(w - 2\sqrt{\frac{2}{g}} \right)}, \quad (5.20)$$

which coincides with (5.12). Thus, $F(w)$ (and consequently, the eigenvalue density of $\phi^\dagger\phi$) is also continuous at the transition.

It is amusing to note that the various expressions we encountered in this subsection are much simpler than their counterparts in the disk phase.

6 Boundaries and Shape Universality of the Eigenvalue Distribution: the Single Ring Theorem

In the previous section we analyzed the probability distribution (1.1) with a quartic potential $V(\phi^\dagger\phi) = m^2\phi^\dagger\phi + \frac{g}{2}(\phi^\dagger\phi)^2$. We obtained circularly symmetric eigenvalue distributions: a disk (5.8) and an annulus (5.18). Making m^2 negative enough drove a phase transition from the disk configuration to the annulus configuration.

This state of affairs obviously generalizes. In this class of models, the domain of the eigenvalue distribution is always circularly symmetric, due to the rotational invariance of the probability distribution (1.1). Consider a potential $V(\phi^\dagger\phi)$ with several wells or minima. For deep enough wells, we expect the eigenvalues of $\phi^\dagger\phi$ to “fall into the wells”. Thus, one might suppose that the eigenvalue distribution of ϕ to be bounded by a set of concentric circles of radii $0 \leq r_1 < r_2 < \dots < r_{n_{\max}}$, separating annular regions on which $\rho(r) > 0$ from voids (annuli in which $\rho(r) = 0$.) A priori, it is natural to assume that the maximal number of such circular boundaries should grow with the degree of V , because V may then have many deep minima.

Remarkably, however, we prove below that the number of these boundaries is two at the most. To reconcile this conclusion with the a priori expectation just mentioned, we note that while the eigenvalues of the hermitean matrix $\phi^\dagger\phi$ may split into several disjoint segments along the positive real axis, this does not necessarily constrain the eigenvalues of ϕ itself to condense into annuli. This statement is made more precise in the Appendix.

We call this result the “Single Ring Theorem.” To prove it, let us assume for the moment that there are n boundaries. It is easy to see from (3.7), the defining equation of $\gamma(r)$, that in the annular void $r_k < r < r_{k+1}$ in the eigenvalue distribution, γ is a constant which is equal to the fraction of eigenvalues contained inside the disk $r \leq r_k$. Thus, the equation to determine γ (Eq. (4.21) which we repeat here for convenience)

$$\gamma \left[r^2 F \left(\frac{\gamma r^2}{\gamma - 1} \right) - \gamma + 1 \right] = 0 \quad (6.1)$$

must have a series of monotonically increasing constant solutions $\gamma_1 < \gamma_2 < \dots \leq 1$,

which correspond to the various voids.

In particular, from (3.9) we have $\gamma = 1$ for $r > r_{n_{\max}}$, namely, $G = 1/z$. Thus, $\gamma = 1$ must be a solution of (6.1), the maximal allowed constant solution. It is straightforward to see that $\gamma = 1$ is indeed a solution, because we know that

$$F\left(\frac{\gamma}{\gamma-1} r^2\right) \sim \frac{\gamma-1}{\gamma r^2}$$

as γ tends to 1. Also, if $r_1 > 0$, namely, if there is a hole at the center of the eigenvalue distribution, then for $r \leq r_1$ we must have $\gamma = 0$ (independently of r), which is obviously a solution¹³ of (6.1). (On the other hand, if $r_1 = 0$, so that the eigenvalue distribution includes the origin, an r independent solution $\gamma = 0$ is of course only a spurious solution which should be discarded.)

Assume now that $\gamma = \gamma_0$ is an r independent solution of (6.1). Taking the derivative of (6.1) with respect to r^2 at $\gamma \equiv \gamma_0$ we obtain

$$\frac{\gamma_0}{\gamma_0-1} \left[F(\xi^2) + \xi^2 \frac{dF}{d\xi^2} \right]_{\xi^2 = \frac{\gamma_0}{\gamma_0-1} r^2} = 0, \quad (6.2)$$

which is the condition for the existence of an r independent solution γ_0 . Thus, there are two possibilities: either $\gamma_0 = 0$, which we already encountered, or $F(\xi^2) + \xi^2 \frac{dF}{d\xi^2} = 0$. This equation immediately yields

$$F(\xi^2) = \frac{1}{\xi^2}, \quad (6.3)$$

where the integration constant is fixed by the asymptotic behavior of $F(\xi^2)$ as $\xi \rightarrow \infty$. But for a generic $V(\phi^\dagger \phi)$, $F(w)$ is given by (Eq. (4.7))

$$F(w) = \left\langle \frac{1}{N} \text{tr}_{(N)} \frac{1}{w - \phi^\dagger \phi} \right\rangle \equiv \int_0^\infty \frac{\tilde{\rho}(\sigma) d\sigma}{w - \sigma},$$

with (6.3) being the asymptotic behavior of $F(w)$ as $w = \xi^2 \rightarrow \infty$. We thus conclude that $\xi^2 \rightarrow \infty$ in (6.3), namely, that $\gamma_0 = 1$. Thus, to summarize, the only possible r independent solutions of (4.21) are $\gamma = 0$ and $\gamma = 1$, which we already discussed.

¹³From the paragraph right below (4.24) it is clear that $wF(w) \rightarrow 0$ at $w = 0$.

Since no other r independent solutions arise, there can be no more than a single void in the eigenvalue distribution, whatever polynomial the potential $V(\phi^\dagger\phi)$ is. The shape of the eigenvalue distribution is thus either a disk or an annulus.

Interestingly enough, we can arrive at the same conclusion by invoking other general aspects of the method of hermitization [4], and thus providing a nice self consistency check of our formalism. Recall at this point that the boundaries of the eigenvalue distribution are given by (3.15), namely, the zeros of \mathcal{G} . But \mathcal{G} is given by (Eq. (4.15))

$$\mathcal{G}^2 = \frac{\gamma(\gamma - 1)}{r^2},$$

from which we see that at the boundaries γ can take on only two values : zero and one. Since γ is a constant in the void, by continuity, these are the only possible values of γ inside any of the voids. Therefore, there may be two circular boundaries at most. In other words, as far as the eigenvalue density of ϕ is concerned, an ensemble of the form (1.1) may have two phases at most, as we concluded earlier.

In addition to $\gamma = 1$ (and possibly $\gamma = 0$), (6.1) must have other roots which do depend on r . Among all these other roots, we expect to find a unique root $\gamma(r)$, which is a positive monotonically increasing function of r , that matches continuously at the boundaries r_k ($k \leq 2$) to the constant roots of (6.1). The actual $zG(z, z^*)$ of the ensemble (1.1) is therefore a continuous monotonically increasing function of r , as required by (3.7). If the eigenvalue distribution is annular with boundaries $0 \leq r_1 < r_2$ (the case $r_1 = 0$ corresponding to the full disk), it vanishes for $0 \leq r \leq r_1$, rises monotonically from zero to one on $r_1 \leq r \leq r_2$, and equals to one for $r \geq r_2$.

7 The Addition Formalism for non-Hermitian Random Matrices

Using the method of hermitization[4], we develop in this section the addition formalism for non-hermitean random matrices. Our results agree with recent results of Zahed *et al.*[2]. We feel that the method of hermitization clearly elucidates the development of this addition formalism. At the end of this section we use the addition formalism to formulate and prove a non-hermitean non-Gaussian version of the central limit theorem.

Let ϕ_1 and ϕ_2 be two $N \times N$ random non-hermitean matrices, drawn independently from the probability distributions $P_1(\phi_1^\dagger \phi_1)$ and $P_2(\phi_2^\dagger \phi_2)$. Furthermore, let H_a ($a = 1, 2$) be the $2N \times 2N$ hermitean Hamiltonians associated with ϕ_a , as defined by (3.10), and similarly define the “total Hamiltonian” as

$$H_T = \begin{pmatrix} 0 & \phi_1 + \phi_2 - z \\ \phi_1^\dagger + \phi_2^\dagger - z^* & 0 \end{pmatrix}. \quad (7.1)$$

The addition problem for ϕ_1 and ϕ_2 may be formulated as follows: Consider the probability distributions

$$P_a(\phi_a^\dagger \phi_a) = \frac{1}{Z_a} \exp \left[-N \text{tr} V_a(\phi_a^\dagger \phi_a) \right], \quad a = 1, 2. \quad (7.2)$$

Given the two propagators

$$\mathcal{G}_\nu^{(a)\mu}(\eta; z, z^*) = \left\langle \left(\frac{1}{\eta - H_a} \right)_\nu^\mu \right\rangle^{(a)} \quad (7.3)$$

(where $\langle \dots \rangle^{(a)}$ indicates an average with respect to P_a), find (in the large N limit) the propagator

$$\mathcal{G}_\nu^\mu(\eta; z, z^*) = \left\langle \left(\frac{1}{\eta - H_T} \right)_\nu^\mu \right\rangle \quad (7.4)$$

associated with the probability distribution

$$P(\phi_1^\dagger \phi_1, \phi_2^\dagger \phi_2) = P_1(\phi_1^\dagger \phi_1) P_2(\phi_2^\dagger \phi_2). \quad (7.5)$$

The propagators $\mathcal{G}_\nu^{(a)\mu}(\eta; z, z^*)$ may be obtained in principle, by applying the formalism of Section 3. Once one has succeeded in solving the addition problem and obtained an explicit expression for (7.4), one may set $\eta = 0$ and calculate the various averages $F(z, z^*) = \langle \frac{1}{N} \text{tr} \log (z - \phi)(z^* - \phi^\dagger) \rangle$, $G(z, z^*) = \langle (1/N) \text{tr} 1/(z - \phi) \rangle$ and $\mathcal{G}(0; z, z^*)$ as prescribed in Section 2, from which one may further calculate the averaged density of eigenvalues $\rho(x, y)$ of $\phi_1 + \phi_2$.

The formulation of the addition problem for non-hermitean random matrices ϕ_1, ϕ_2 bears obvious similarities to the formulation of the addition problem for hermitean random matrices. In fact, if in the formulation just given one replaces the bare inverse propagator \mathcal{G}_0^{-1} defined in (4.1) simply by $\mathcal{G}_0^{-1} \equiv z$, and the chiral matrices

$$\begin{pmatrix} 0 & \phi_a \\ \phi_a^\dagger & 0 \end{pmatrix}$$

by the ordinary hermitean matrices φ_a ($a = 1, 2$), one would recover the formulation of the usual addition problem for hermitean random matrices[13, 14, 16]. These identifications of various quantities that appear in the formulations of the two addition problems, enable us to derive the addition formalism for random non-hermitean matrices by following the steps of the by now well known addition formalism for ordinary hermitean random matrices[14], which we now proceed to do.

Let $\Gamma_{2k}^{(a)}$ and

$$\Sigma_\nu^{(a)\mu} = \Sigma^{(a)}(\eta; r) \delta_\nu^\mu \quad (7.6)$$

be, respectively, the connected $2k$ gluon “blob” (or cumulant) and the self-energy associated with the probability distribution $P_a(\phi_a^\dagger \phi_a)$. From the definition of self-energy

$$\mathcal{G}_\nu^\mu = \left(\frac{1}{\mathcal{G}_0^{-1} - \Sigma} \right)_\nu^\mu$$

(Eq. (4.2)) we find

$$\Sigma_\nu^{(a)\mu} = \left(\mathcal{G}_0^{-1} - \mathcal{G}^{-1(a)} \right)_\nu^\mu. \quad (7.7)$$

In addition, rewriting the expansion (4.6) for $\Sigma^{(a)\mu}_\nu$ we have

$$\Sigma^{(a)}(\eta; r) = \sum_{k=1}^{\infty} \Gamma_{2k}^{(a)} \left[\frac{1}{2N} \mathcal{G}^{(a)\mu}_\mu(\eta; z, z^*) \right]^{2k-1} \equiv \sum_{k=1}^{\infty} \Gamma_{2k}^{(a)} \left[\mathcal{G}^{(a)}(\eta; r) \right]^{2k-1}. \quad (7.8)$$

For reasons that will become clear below we bothered to display the trace operation in (7.8) explicitly. The two gluon types ϕ_1 and ϕ_2 in the combined probability distribution (7.5) do not interact and therefore the self-energy $\Sigma^\mu_\nu = \Sigma(\eta; r)\delta^\mu_\nu$ associated with (7.5) is given by

$$\Sigma(\eta; r) = \sum_{k=1}^{\infty} \left(\sum_a \Gamma_{2k}^{(a)} \right) \left[\frac{1}{2N} \mathcal{G}^\mu_\mu(\eta; z, z^*) \right]^{2k-1} \equiv \sum_{k=1}^{\infty} \left(\sum_a \Gamma_{2k}^{(a)} \right) [\mathcal{G}(\eta; r)]^{2k-1}, \quad (7.9)$$

where now obviously the internal quark lines are given by (7.4).

The right hand side of (7.8) and of (7.9) are expansions in powers of the trace of three different propagators: $\mathcal{G}^{(a)\mu}_\nu$ ($a = 1, 2$) and \mathcal{G}^μ_ν . These three propagators are obviously not the same, but we observe that if they were identical, then (7.9) would simply be the sum (over the two values of a) of (7.8). The fact that these three propagators are not identical renders the relation between the three self-energies more complicated. The essence of the addition formalism is to maneuver around this difficulty. The idea is as follows: for the sake of argument, assume that it is mathematically consistent to consider $\mathcal{G}^{(a)\mu}_\nu(\eta; , r)$ as a matrix valued function of the inverse propagator matrix

$$\mathcal{G}_0^{-1} = \begin{pmatrix} \eta & z \\ z^* & \eta \end{pmatrix}$$

(Eq. (4.1).) This function would then have a matrix valued functional inverse \mathcal{B}_a , where $\mathcal{B}_a \left(\mathcal{G} \left(\mathcal{G}_0^{-1} \right) \right) = \mathcal{G}_0^{-1}$. Naively, we may deduce from (7.7) that

$$\mathcal{B}_a(\mathcal{G}^{(a)\alpha}_\beta)_\nu^\mu = \left(\mathcal{G}^{-1(a)} \right)_\nu^\mu + \Sigma^{(a)\mu}_\nu \quad (7.10)$$

(below we shall do away with the adverb “naively” and make the construction of \mathcal{B}_a mathematically precise.) Using this matrix valued functional inverse we would be able to solve the equation $\mathcal{G}^{(a)\mu}_\nu(\mathcal{X}_a) = \mathcal{G}^\mu_\nu$ (where \mathcal{X}_a is the matricial argument of the

matrix valued function $\mathcal{G}^{(a)\mu}_\nu$) as $\mathcal{X}_a = \mathcal{B}_a(\mathcal{G}_\nu^\mu)$. We would then evaluate each of the $\Sigma^{(a)}$ in (7.8) at the appropriate argument \mathcal{X}_a (instead at argument \mathcal{G}_0^{-1}). The right hand side of (7.8) would then be an expansion in powers of $\mathcal{G}(\eta; r)$, and we would thus conclude that $\sum_a \Sigma^{(a)}(\mathcal{X}_a) = \Sigma(\mathcal{G}_0^{-1})$.

The arguments in the last paragraph follow the same logic as in the case of the addition formalism of random hermitean matrices. Recall that in the theory of ordinary hermitean random matrices, the Blues function $\mathcal{B}(w)$ and the Green's function $\mathcal{G}(z) \equiv (1/N)\langle \text{tr} (z - \varphi)^{-1} \rangle$ are functional inverses of one another[14]. In the present context, we already noted that the ordinary complex variable z is replaced by the matrix \mathcal{G}_0^{-1} in (4.1), which leads us to consider matrix valued functions of \mathcal{G}_0^{-1} and composition thereof. Thus, following the nomenclature of the hermitean addition formalism, we name \mathcal{B}_a the “Blue’s” function associated with $\mathcal{G}^{(a)\mu}_\nu$. We now have to complete the arguments in the previous paragraph and show that composition of such matrix valued functions of matricial arguments are mathematically meaningful.

In order that composition of such functions be a consistent operation (in the sense that it generalizes composition of complex-analytic functions of a complex variable), it is necessary that these matrix valued functions (with \mathcal{G}_ν^μ in (4.4) being an important example), have the same matricial texture as \mathcal{G}_0^{-1} (Eq. (4.1)). Recall that (Eq. (4.4))

$$\mathcal{G}_\nu^\mu(\eta; z, z^*) = \frac{1}{r^2 - (\eta - \Sigma)^2} \begin{pmatrix} \Sigma - \eta & z \\ z^* & \Sigma - \eta \end{pmatrix},$$

where η and $\Sigma \equiv \Sigma(\eta; r)$ are complex. Thus, for η complex, the texture of \mathcal{G}_ν^μ is more complicated than the texture of

$$\mathcal{G}_0^{-1} = \begin{pmatrix} \eta & z \\ z^* & \eta \end{pmatrix}$$

(Eq. (4.1), whose off-diagonal blocks are hermitean conjugates of each other. This mismatch of textures can be dealt with readily by generalizing the space of matrices

out of which \mathcal{G}_0^{-1} is taken into

$$\mathcal{G}_0^{-1} = \begin{pmatrix} \eta & z \\ w & \eta \end{pmatrix}. \quad (7.11)$$

Here the off diagonal blocks are no longer related. To see that this generalization is the natural one to make, we follow the first few steps of analysis in Section 4 and find that with (7.11) the self-energy matrix is

$$\Sigma_\nu^\mu = \Sigma(\eta; zw) \delta_\nu^\mu, \quad (7.12)$$

(where the only change in (4.3) is that now Σ depends on zw rather than on $r^2 = z^*z$.) Thus, from the definition of self-energy (4.2) we immediately see that (4.4) is changed into

$$\mathcal{G}_\nu^\mu(\eta; z, w) = \frac{1}{zw - (\eta - \Sigma)^2} \begin{pmatrix} \Sigma - \eta & z \\ w & \Sigma - \eta \end{pmatrix} \equiv \begin{pmatrix} \mathcal{G} & G \\ G' & \mathcal{G} \end{pmatrix}. \quad (7.13)$$

This has essentially the same texture as (4.4). With this extension, we have gained the fact that (7.11), which generalizes here the ordinary complex variable z , has the same matricial texture as (7.13). At the end of our calculations we will set $w = z^*$. We can also see that the new texture (7.11) is the right one at a more elementary level, by simply recognizing that the texture of (7.11) is preserved by raising this new \mathcal{G}_0^{-1} to any integer (positive or negative) power.

With this texture it is enough to consider \mathcal{G}_a and \mathcal{G}_0^{-1} simply as 2×2 complex matrices with equal diagonal elements. Viewing \mathcal{G}_a ($a = 1, 2$) as a 2×2 matrix valued function of a matricial argument (7.11) is an equivalent way of saying that we are dealing with the three functions $\mathcal{G}_a(\eta; zw)$, $G_a(\eta; z, w)$ and $G'_a(\eta; z, w)$, and composition of such matrix valued functions amounts to compositions of such triads in the obvious way.

In the following the symbols $\mathcal{B}, \mathcal{W}, \mathcal{X}$ etc. will stand for matrices of a texture identical to that of \mathcal{G}_0^{-1} and \mathcal{G}_a . The matrix valued Blue's function $\mathcal{B}_a(\mathcal{W})$, associated

with the probability distribution $P_a(\phi_a^\dagger \phi_a)$, ($a = 1, 2$) is defined then as the matrix

$$\mathcal{B}_a = \begin{pmatrix} B_a & b_a \\ b'_a & B_a \end{pmatrix}, \quad (7.14)$$

that satisfies

$$\mathcal{B}_a [\mathcal{G}_a (\mathcal{X})] = \mathcal{X}. \quad (7.15)$$

Similarly, we define the 2×2 matrix “Blue’s” function \mathcal{B} associated with (7.5) by $\mathcal{B} [\mathcal{G} (\mathcal{X})] = \mathcal{X}$. From the definition of self-energy (4.2), or equivalently from (7.7), we readily identify

$$\mathcal{B}(\mathcal{W}) = \mathcal{W}^{-1} + \mathbf{1}\Sigma(\mathcal{W}) = \mathcal{W}^{-1} + \mathbf{1} \sum_{k=1}^{\infty} \Gamma_{2k} \left(\frac{1}{2} \text{tr} \mathcal{W} \right)^{2k-1}, \quad (7.16)$$

where $\mathbf{1}$ is the 2×2 unit matrix.

The identity of textures on both sides of (7.15) is assured already by the definitions of the quantities involved, and does not pose by itself any constraints on the various matrix elements that appear on both sides of (7.15). In other words, \mathcal{B}_a is indeed the matrix valued inverse function of the matrix valued function \mathcal{G}_a . Furthermore, due to this equality of textures we can readily apply \mathcal{G}_a to both sides of (7.15) obtaining

$$\mathcal{G}_a [\mathcal{B}_a [\mathcal{G}_a (\mathcal{X})]] = \mathcal{G}_a (\mathcal{X}), \quad (7.17)$$

which verifies that the matricial function composition in (7.15) respects the obvious condition that taking the functional inverse twice in a row yields the original function. Thus, setting for example $\mathcal{X} = \mathcal{G}_0^{-1}$ in (7.15), that equation simply says that starting with the set of three functions $\mathcal{G}_a(\eta; zw)$, $G_a(\eta; z, w)$ and $G'_a(\eta; z, w)$, we construct the set of three functions $B_a(\eta; zw)$, $b_a(\eta; z, w)$ and $b'_a(\eta; z, w)$, such that $\mathcal{G}_a(B_a; b'_a b_a) = \eta$, $G_a(B_a; b_a, b'_a) = z$ and $G'_a(B_a; b_a, b'_a) = w$.

Note from (7.16) that $\mathcal{B}(-\mathcal{W}) = -\mathcal{B}(\mathcal{W})$. This is so because \mathcal{B} is the functional inverse of the propagator $\mathcal{G}(\mathcal{X})$ which is manifestly an odd function of \mathcal{X} due to the fact that in this paper all the probability distributions are even in ϕ .

Let us now go back to (7.8). The propagator $\mathcal{G}_\nu^{(a)\mu}$ on the right hand side of (7.8) is a function of η, z and w , namely, of the matrix \mathcal{G}_0^{-1} . We may generalize and consider it a function of the variable \mathcal{W} , rendering $\Sigma^{(a)}$ a function of \mathcal{W} , namely, $\Sigma^{(a)}(\mathcal{W}) = \sum_{k=1}^{\infty} \Gamma_{2k}^{(a)} \left[\frac{1}{2} \text{tr } \mathcal{G}^{(a)}(\mathcal{W}) \right]^{2k-1}$ as in (7.16). Similarly, by replacing the argument \mathcal{G}_0^{-1} of \mathcal{G}_ν^μ on the right hand side of (7.9) by a variable \mathcal{X} (with \mathcal{X} having the same texture as all the matrices above), we may consider the function $\Sigma(\mathcal{X}) = \sum_{k=1}^{\infty} \Gamma_{2k}^{(a)} \left[\frac{1}{2} \text{tr } \mathcal{G}(\mathcal{X}) \right]^{2k-1}$.

Evaluating $\Sigma^{(a)}(\mathcal{W})$ at $\mathcal{W} = \mathcal{W}_a \equiv \mathcal{B}_a [\mathcal{G}(\mathcal{G}_0^{-1})]$, and using (7.15) and (7.9) we immediately find that

$$\sum_a \Sigma^{(a)}(\mathcal{W}_a) = \sum_{k=1}^{\infty} \left(\sum_a \Gamma_{2k}^{(a)} \right) [\mathcal{G}(\eta; r)]^{2k-1} \equiv \Sigma(\eta; r). \quad (7.18)$$

Thus, from (7.16) and from the matrix inverse of (7.15) we obtain

$$\sum_{a=1,2} \left\{ \mathcal{B}_a [\mathcal{G}(\mathcal{X})] - \mathcal{G}^{-1}(\mathcal{X}) \right\} = \mathcal{X} - \mathcal{G}^{-1}(\mathcal{X}),$$

namely,

$$\sum_{a=1,2} \mathcal{B}_a [\mathcal{G}(\mathcal{X})] - \mathcal{G}^{-1}(\mathcal{X}) = \mathcal{X}. \quad (7.19)$$

Therefore, by definition, the Blue's function associated with (7.5) is simply

$$\mathcal{B}_T(\mathcal{W}) = \mathcal{B}_1(\mathcal{W}) + \mathcal{B}_2(\mathcal{W}) - \mathcal{W}^{-1}, \quad (7.20)$$

in agreement with [2].

To summarize, we carried out diagrammatic expansions (namely, expansions in traces of powers of $\mathcal{G}_\nu^{(a)\mu}$ and \mathcal{G}_ν^μ) of the self-energies $\Sigma^{(a)}$ and Σ . We were allowed to do so because by construction \mathcal{G}_0^{-1} was taken to be non-hermitean, and thus in the analyticity domain of \mathcal{G}_ν^μ . Those diagrammatic expansions led us to introduce the matrix valued Blue's functions (7.14), in close analogy with the addition formalism for ordinary hermitean random matrices [14]. As in the latter case, the Blue's functions follow a simple addition formula (7.20) under multiplication of the individual probability distributions (7.5).

Now that we have \mathcal{B}_T in (7.20), we can calculate the Green's function $\mathcal{G}_\nu^\mu(\eta; z, w)$ associated with H_T by evaluating the functional inverse of \mathcal{B}_T in (7.20) at argument \mathcal{G}_0^{-1} . After completing this step we may return to the original conventions and set $w = z^*$. Also, for the sole purpose of determining the eigenvalue distribution of $\phi = \phi_1 + \phi_2$ all we need is $\mathcal{G}_\nu^\mu(0 - i0; z, z^*)$, and we may also set $\eta = 0 - i0$.

7.1 A Simple Application: A Non-abelian Non-Hermitean Central Limit Theorem

Gauss proved that if we add K random numbers x_i , $i = 1, 2, \dots, K$, with x_i taken from the probability distribution $P_i(x_i)$, then the normalized sum $s = \frac{1}{\sqrt{K}} \sum_i x_i$ follows the Gaussian distribution in the limit K tending to infinity. This result plays an important role in physics and mathematics and accounts for the ubiquitous appearance of the Gaussian distribution.

A generalization of Gauss's central limit theorem for the case in which the variables x_i are replaced by $N \times N$ hermitean random matrices φ_i , $i = 1, 2, \dots, K$, taken from from the probability distributions

$$P_i(\varphi_i) = \frac{1}{Z_i} e^{-N \text{tr} V_i(\varphi_i)}. \quad (7.21)$$

certainly exists and was recently discussed in [14].¹⁴ In this version of the theorem the normalized sum $\varphi_T \equiv \frac{1}{\sqrt{K}} \sum_i \varphi_i$ indeed follows the Gaussian distribution in the large K limit.

In this sub-section we wish to discuss Gauss's theorem for $N \times N$ non-hermitean random matrices ϕ_i , $i = 1, 2, \dots, K$, that are taken from from the probability distributions

$$P_i(\phi_i^\dagger \phi_i) = \frac{1}{Z_i} e^{-N \text{tr} V_i(\phi_i^\dagger \phi_i)}. \quad (7.22)$$

¹⁴Note that there exists another commonly considered class of random matrices, in which the element of the random matrices is each taken from a probability distribution (the same for each element). For this class, which is sometimes referred to as the Wigner distribution[6], the theorem follows immediately from the usual abelian central limit theorem. Here we are speaking of the trace classes defined by (7.21).

As was already remarked in [14], it is not difficult to generalize one of the standard proofs of Gauss's theorem to matrices, and we shall do so at the end of this subsection. However, we find it quite interesting to address the question of whether $\phi_T \equiv \frac{1}{\sqrt{K}} \sum_i \phi_i$ follows the Gaussian distribution by applying the non-hermitean addition formalism just presented. In doing so, we shall generalize the analysis made in the last section of [14] to non-hermitean random matrices and show that Ginibre's disk distribution [10] for the eigenvalues of ϕ_T emerges naturally from the addition formalism.

It is clear from the derivation of the addition formalism just given that when we add two random matrices, in general it is difficult to determine explicitly the resulting \mathcal{G}_ν^μ for the sum of the two matrices. What we hope for here is that the large K limit will bring considerable simplification. This is indeed the case.

From the law of addition (7.20) we learn that the function $\mathcal{B}(\mathcal{W})$ associated with the unknown $\mathcal{G}(\mathcal{X})$ is given by

$$\mathcal{B}(\mathcal{W}) = \sum_{i=1}^K \mathcal{B}_a(\mathcal{W}) - \frac{K-1}{\mathcal{W}}. \quad (7.23)$$

Using (7.16) we may rewrite (7.23) as

$$\mathcal{B}(\mathcal{W}) = \sum_{k=1}^{\infty} \left(\sum_{i=1}^K \Gamma_{2k}^{(i)} \right) \left(\frac{1}{2} \text{tr} \mathcal{W} \right)^{2k-1} + \frac{1}{\mathcal{W}}, \quad (7.24)$$

where $\Gamma_{2k}^{(i)}$ is the $2k$ -th cumulant associated with the distribution P_i in (7.22).

By definition (Eq. (7.14)), the Green's function $\mathcal{G}(\mathcal{X})$ associated with $\sum_i \phi_i$ is determined by solving $\mathcal{B}(\mathcal{G}(\mathcal{X})) = \mathcal{X}$. We are however interested in the normalized sum $\phi_T \equiv \frac{1}{\sqrt{K}} \sum_i \phi_i$. Define the corresponding propagator as

$$\mathcal{G}_T(\mathcal{X}) \equiv \left\langle \left(\frac{1}{\mathcal{X} - \frac{1}{\sqrt{K}} \sum_i \mathcal{H}_i} \right) \right\rangle = \sqrt{K} \mathcal{G}(\sqrt{K} \mathcal{X}), \quad (7.25)$$

where

$$\mathcal{H}_i = \begin{pmatrix} 0 & \phi_i \\ \phi_i^\dagger & 0 \end{pmatrix}.$$

Rescaling and substituting $\mathcal{G}(\sqrt{K}\mathcal{X})$ into (7.24), we find that $\mathcal{G}_T(\mathcal{X})$ is determined by

$$\frac{1}{\mathcal{G}_T(\mathcal{X})} + \sum_{k=1}^{\infty} \left[\frac{1}{K^k} \left(\sum_{i=1}^K \Gamma_{2k}^{(i)} \right) \left(\frac{1}{2} \text{tr} \mathcal{G}_T(\mathcal{X}) \right)^{2k-1} \right] = \mathcal{X}. \quad (7.26)$$

For finite K , (7.26) is hopelessly complicated. However, as $K \rightarrow \infty$ we see that it simplifies rather naturally to

$$\mathcal{G}_T^{-1}(\mathcal{X}) + \frac{\Gamma_2}{2} \text{tr} \mathcal{G}_T(\mathcal{X}) = \mathcal{X} \quad (7.27)$$

with

$$\Gamma_2 \equiv \frac{1}{K} \sum_i^K \Gamma_2^{(i)}. \quad (7.28)$$

The self-energy associated with (7.27) is evidently $\Sigma_T = \frac{\Gamma_2}{2} \text{tr} \mathcal{G}_T(\mathcal{X})$. It is the simplest case of the general formula (Eq.(7.8)) $\Sigma(\mathcal{G}) = \sum_{k=1}^{\infty} \Gamma_{2k} \left(\frac{1}{2} \text{tr} \mathcal{G} \right)^{2k-1}$, where only $\Gamma_2 \neq 0$. In other words, it is equal to the self-energy associated with the Gaussian distribution

$$P_G(\phi^\dagger \phi) = \frac{1}{Z} \exp \left(-\frac{N}{\Gamma_2} \text{tr} \phi^\dagger \phi \right). \quad (7.29)$$

Re-substituting

$$\mathcal{X} = \mathcal{G}_0^{-1} = \begin{pmatrix} \eta & z \\ z^* & \eta \end{pmatrix},$$

and using the general hermitization formalism in Section 2 (and in particular Eqs. (3.3) and (3.12)) we conclude [2, 4] that the eigenvalues of $\phi_T = (1/\sqrt{K}) \sum_i^K \phi_i$ are distributed uniformly in a disk of radius $\sqrt{\Gamma_2}$.

Thus, not only do we obtain Ginibre's distribution, we also learn that Γ_2 is given by (7.28).

While we have proved that the density of eigenvalues of ϕ_T is identical to the density associated with the Gaussian distribution (7.29), we cannot yet conclude that the probability distribution of ϕ_T is Gaussian. The reason is that in principle, a given eigenvalue distribution may correspond to more than a single probability

distribution¹⁵. To see that ϕ_T is indeed taken from a Gaussian distribution, we have to calculate that distribution explicitly.

As mentioned above, it is not difficult to extend one of the usual proofs of the central limit theorem to the case of matrices. The distribution for the normalized sum matrix $\phi \equiv \frac{1}{\sqrt{K}} \sum_i \phi_i$ is given by (here we omit the subscript T)

$$\begin{aligned} P(\phi) &= \left(\prod_i^K \int d\phi_i P_i(\phi_i^\dagger \phi_i) \right) \delta \left(\phi - \frac{1}{\sqrt{K}} \sum_i \phi_i \right) \\ &= \int dt \left(\prod_i^K \int d\phi_i P_i(\phi_i^\dagger \phi_i) \right) e^{\frac{i}{\sqrt{K}} \sum_i^K \text{Im tr } t\phi_i} e^{-i \text{Im tr } t\phi}. \end{aligned} \quad (7.30)$$

The integral over the ϕ_i can be done in the large K limit:

$$\begin{aligned} &\int d\phi_i P(\phi_i) e^{\frac{i}{\sqrt{K}} \sum_i \text{Im tr } t\phi_i} \\ &= 1 - \frac{1}{4K} \int d\phi_i P(\phi_i^\dagger \phi_i) \text{tr}(t\phi_i) \text{tr}(t^\dagger \phi_i^\dagger) + \mathcal{O}\left(\frac{1}{K^2}\right) \\ &= 1 - \frac{\Gamma_2^{(i)}}{4KN} \text{tr}(t^\dagger t) + \mathcal{O}\left(\frac{1}{K^2}\right) \end{aligned} \quad (7.31)$$

where by definition $\Gamma_2^{(i)} = \int d\phi_i P(\phi_i^\dagger \phi_i) \frac{1}{N} \text{tr} \phi_i^\dagger \phi_i$. Reexponentiating and integrating over t we obtain the desired result that $P(\phi)$ is proportional to $e^{-\frac{N}{\Gamma_2} \text{tr} \phi^\dagger \phi}$, with Γ_2 given by (7.28).

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¹⁵This is indeed the case for hermitean random matrices. For example, for matrices in the Wigner class, even when the distribution is not Gaussian, the corresponding density of eigenvalues still satisfies the semi-circle law, as is well known.

for performing numerical checks of this work, which uncovered an error in Section 5.1.

Appendix : A Note Concerning the “Single Ring Theorem”

The hermitean matrix $\phi^\dagger\phi$ can always be diagonalized $\phi^\dagger\phi = U^\dagger\Lambda^2U$ by a unitary matrix U , with $\Lambda^2 = \text{diag}(\lambda_1^2, \lambda_2^2, \dots, \lambda_N^2)$, where the λ_i are all real. This implies that $\phi = V^\dagger\Lambda U$, with V a unitary matrix as well. Thus, the complex eigenvalues of ϕ are given by the roots of $\det(z - \Lambda W) = 0$, with $W = UV^\dagger$. Evidently, as W ranges over $U(N)$, the eigenvalues of ΛW could be smeared (in the sense that they would not span narrow annuli around the circles $|z| = |\lambda_i|$.) In Section 6, we proved (in the large N limit) that for ϕ taken from the class of ensembles of the form $P(\phi^\dagger\phi) = (1/Z)\exp[-N\text{tr } V(\phi^\dagger\phi)]$ (Eq. (1.1)), the eigenvalues can only span either a disk or an annulus. We have verified that this is the case, explicitly for $N = 2$, and numerically for large N (of the order of 100) and for some range of Λ^2 .

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